Faculty of Science, Technology, Engineering and Mathematics
M303 Further pure mathematics

M303

TMA 02 2020J

Mainly covers Book B *Groups*
Cut-off date 15 December 2020

You will find instructions for completing TMAs in the 'Assessment' area of the M303 website. Please remind yourself of these instructions before beginning work on this TMA. You can submit this TMA either by post, together with a completed TMA form (PT3), to arrive by the cut-off date given above, or electronically, as a single PDF file, using the University's online TMA/EMA service.

Half of the TMA is formative, and the other half is summative. The formative questions are designed to either *extend* your understanding of the topics or *introduce* the topics of later books, while the summative questions place more emphasis on *assessing* your understanding of the topics. Both types of question will be marked by your tutor, but (unlike the summative questions) marks for the formative questions do not count towards your final grade. You should submit your solutions to both sets of questions to your tutor by the cut-off date.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on this assignment require numerical answers, the questions invariably carry *method marks*.

In phrasing the questions, we have used the words listed below, each of which should be interpreted as indicated.

prove, show, explain, justify	Clear reasoning and explanation for all steps are called for.
determine, find, devise, calculate, compute	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, give, write down, list	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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This TMA is based on Book B *Groups*, except for the formative Question 8, which covers the start of Book C.

Part A is **summative**. These questions assess your knowledge of the module. You must not discuss these questions in the forums, and your tutor is not allowed to help you directly.

Part B is **formative**. You should choose two formative questions to submit to your tutor for marking, and we suggest that one of these is Question 8 since this will help you with the next TMA.

Doing the formative questions will help you to understand key concepts covered in the text. You are encouraged to discuss these questions in the module forums and with your tutor (who is allowed to help you).

Question 8 is designed to help you with TMA 03. Questions 9, 10 and 11 cover material that is more central to the module, while Question 12 is designed to expand your understanding beyond the core material.

Your tutor will mark all of the answers that you submit. But note that only your marks for Part A count towards your overall continuous assessment score.

Part A Summative

Question 1 – 3 marks

This question has two parts. Each part requires you to use the M303 website. If, for any reason, you cannot access the website, then contact your tutor.

(a) Look at the Week 6 (7–14 November) section of the M303 Study Calendar.

(i) Two sections of Chapter 6 are described as core. Write down the numbers of these two core sections of Chapter 6.

(ii) One section of Chapter 6 is described as non-essential. Write down the number of this non-essential section of Chapter 6. [*1*]

(b) Attempt Practice Quiz 2, and submit your attempt. In your TMA, include a screenshot or photo of the page that appears when you have submitted your attempt.

(It does not matter what your score is. Full marks for this question part can be obtained for any *attempt* at the quiz. If you cannot access the quiz for any reason, then contact your tutor.) [*2*]

Question 2 – 13 marks

This question tests your understanding of material covered in Chapter 5.

Let G be the set given by $G = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$. Let $+_G$ be the binary operation on G given by

$$(a + b\sqrt{-3}) +_G (a' + b'\sqrt{-3}) = (a + a') + (b + b')\sqrt{-3},$$

for $a, a', b, b' \in \mathbb{Z}$.

Let $H = \{a + b\sqrt{-3} : a, b \in 2\mathbb{Z}\}$ (where $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ is the set of multiples of 2).

(a) Write down the following.

(i) An identity element of G.

(ii) An element of H that is not the identity.

(iii) An element of G that is not in H.

(iv) The sum (under $+_G$) of your answers to parts (a)(ii) and (a)(iii). [*2*]

(b) Show that $(G, +_G)$ is a group. [*4*]

(c) By using the subgroup criterion, or otherwise, show that H is a subgroup of G. [*2*]

(d) Explain why H is normal. [*1*]

(e) List all the elements of the quotient group G/H, and write out the Cayley table. [*3*]

(f) Is G/H cyclic? [*1*]

Question 3 – 3 marks

This question tests your understanding of material covered in Chapter 5.

This question concerns the additive group of rational numbers, that is, $(\mathbb{Q}, +)$, where $+$ is the usual addition of rational numbers.

Let ϕ be the homomorphism from $\mathbb{Q}$ to $\mathbb{Q}$ given by $\phi(q) = 2q/3$. You do not have to prove that ϕ is a homomorphism.

(a) Write down the kernel of ϕ. [*1*]

(b) Prove that ϕ is onto. [*1*]

(c) Is ϕ an isomorphism? [*1*]

Question 4 – 12 marks

This question tests your understanding of material covered in Chapter 6.

(a) Let $G = \mathbb{Z}_7 \times \mathbb{Z}_{12}$.

(i) Show that $(3, 1)$ generates G. [2]

(ii) Show that $(1, 3)$ does not generate G. [2]

(b) Consider the following abelian groups of order 12,600:

$$\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_{175}, \quad \mathbb{Z}_4 \times \mathbb{Z}_{14} \times \mathbb{Z}_{45} \times \mathbb{Z}_5, \quad \mathbb{Z}_{30} \times \mathbb{Z}_6 \times \mathbb{Z}_{70}.$$

(i) For each of these groups, write down a direct product of cyclic groups of prime power order that is isomorphic to the group. [3]

(ii) Determine which, if any, of the groups is cyclic. [3]

(c) Write down two different (non-isomorphic) non-cyclic abelian groups of order 3213. [2]

Question 5 – 12 marks

This question tests your understanding of material covered in Chapters 6 and 7.

Let D_{30} be the dihedral group

$$D_{30} = \langle r, s \mid r^{30} = s^2 = e,\ sr = r^{29}s \rangle.$$

You may assume that each element can be expressed uniquely in the standard form $r^i s^j$ for $i = 0, \ldots, 29$, $j = 0, 1$, and that $sr^{-i} = r^i s$.

(a) Using the subgroup criterion (or otherwise), show that $H = \{r^{2i}s^j : i = 0, \ldots, 8,\ j = 0, 1\}$ is a subgroup of D_{30}. [3]

(b) By considering the index of H (or otherwise), show that the subgroup H in part (a) is a normal subgroup of D_{30}. [1]

(c) Let K be the subgroup generated by the element r^{15}. By showing that $|K| = 2$, deduce that $K \cong \mathbb{Z}_2$. [1]

(d) By writing sr^{15} in standard form (or otherwise), prove that K is contained in the centre of D_{30}, and deduce that K is normal in D_{30}. [3]

(e) Show that $H \cap K = \{e\}$. [1]

(f) Hence, by using Proposition 1.9 of Chapter 6 (or otherwise), deduce that D_{30} is isomorphic to the direct product $H \times K$. [3]

Question 6 – 4 marks

This question tests your understanding of material covered in Chapter 8.

Let G be a group of order 3179.

(a) State the orders of the Sylow p-subgroups of G. [2]

(b) Use the Sylow Theorems to determine the possible answers to the question: 'How many Sylow p-subgroups can G have?' [2]

Question 7 – 3 marks

Let $p \geq 5$ be a prime. Consider the following 'proof' that any group of order $2p$ is abelian.

(Incorrect) 'proof'

Line 1 Let $p \geq 5$ be a prime number, and let G be a group of order $2p$.

Line 2 Then G has a subgroup of order 2 and a subgroup of order p.

Line 3 Let H be the subgroup of order 2, and let K be the subgroup of order p.

Line 4 Since H has order 2, it is isomorphic to the cyclic group $\mathbb{Z}_2$.

Line 5 Since K has prime order p, it is isomorphic to the cyclic group $\mathbb{Z}_p$.

Line 6 Since $H \cong \mathbb{Z}_2$ and $K \cong \mathbb{Z}_p$ are both cyclic, they are both also abelian.

Line 7 Since H and K are both abelian subgroups of G, they are both normal subgroups of G.

Line 8 The subgroups H and K have coprime orders, and $|G| = 2 \times |\mathbb{Z}_p|$.

Line 9 Therefore $G \cong H \times K$ (by Theorem 2.3 of Chapter 8), where H and K are abelian and so G is also abelian.

(a) Write down a counter-example to this 'proof'. (That is, write down – by name or presentation – a non-abelian group that has order equal to $2p$ for a prime $p \geq 5$). [*1*]

(b) Line 9 of the 'proof' would be true *if* the first 8 lines were all correct. They are not all correct. One of them is incorrect. Write down the line number of the incorrect line, and explain briefly why the line is incorrect. [*2*]

Part B Formative

Question 8 – 9 marks

This question concerns Chapter 9 of Book C, and relates to Question 1 in TMA 03.

(a) Suppose that a positive integer n is such that $4n + \sigma(n)$ is divisible by 5.

(i) Show that n cannot be prime. [2]

(ii) If $n = p^2$, where p is a prime number, show that $p \equiv 4 \pmod 5$. [2]

(iii) Given an example of a number $n > 1$ such that $4n + \sigma(n)$ is divisible by 5. [1]

(b) Determine the pairs of distinct primes p, q for which $pq + \sigma(pq)$ is even. [2]

(c) Suppose that an odd positive integer m is such that $\tau(m)$ divides m. Show that m is a perfect square. [2]

Question 9 – 8 marks

This question will help your understanding of material covered in Chapter 5.

Let $C[0, 1]$ be the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$. Define an operation $+$ on this set by

$f + g$ is the function from $[0, 1]$ to $\mathbb{R}$ given by $(f + g)(x) = f(x) + g(x)$.

For example, if $f : [0, 1] \to \mathbb{R}$ is the continuous function such that $x \mapsto 2x$, and $g : [0, 1] \to \mathbb{R}$ is the constant function such that $x \mapsto -3$, then $f + g : [0, 1] \to \mathbb{R}$ is the function such that $x \mapsto 2x - 3$.

Show that $(C[0, 1], +)$ is a group. You may assume that addition in $\mathbb{R}$ is associative. [8]

Question 10 – 9 marks

This question is intended to offer some practice with quotient groups. The notion of quotient is important, and will appear again for rings and fields in Book E. Quotient groups are covered in Chapter 5, but note that more details on quotients of dihedral groups are given in Chapter 7.

The dihedral group of order 24 has presentation

$$D_{12} = \langle r, s \mid r^{12} = s^2 = e,\ sr = r^{11}s \rangle.$$

You may assume that the elements of D_{12} can be written uniquely in the standard form $r^i s^j$ for $i = 0, 1, \ldots, 11$, $j = 0, 1$.

(a) Write down the elements of a cyclic subgroup H of D_{12} of order 6. [1]

(b) Prove that H is a normal subgroup of D_{12}. [4]

(c) (i) List the elements of each of the four cosets H, rH, sH and rsH of the quotient group D_{12}/H, and write down the Cayley table of D_{12}/H. [3]

(ii) Is D_{12}/H cyclic? Justify your answer briefly. [1]

Question 11 – 16 marks

This question involves many of the fundamental concepts from Book B. You might find that you are able to start this question soon after starting Book B, but will not be able to finish it until later in your study of the book.

Consider the groups

$$D_{12}, \quad \mathbb{Z}_{12}, \quad \mathbb{Z}_3 \times \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \mathbb{Z}_2 \times \mathbb{Z}_6, \quad \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_4.$$

(a) Write down the two pairs of isomorphic groups in this list. [2]

(b) Complete the table below (using the groups in the list above). A copy of the table is provided on page 8 of this booklet for you to print off and fill in if you wish.

As an example, the two groups $\mathbb{Z}_{15}$ and $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ have been put into the table below. They are not included in the list of groups that you are asked to work with. [12]

Group	Order of group	Isomorphic groups	Orders of elements	Cyclic? Abelian?
$\mathbb{Z}_{15}$	15	$\mathbb{Z}_3 \times \mathbb{Z}_5$	1, 3, 5, 15	Cyclic
$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	20	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	1, 2, 5, 10	Abelian, non-cyclic
D_{12}				
$\mathbb{Z}_3 \times \mathbb{Z}_4$				
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$				

(c) Write down any groups from the table that have a subgroup isomorphic to $\mathbb{Z}_4$. [2]

Question 12 – 8 marks

This question is intended to deepen your understanding of direct products.

(a) The dihedral group D_{10} is given by

$$D_{10} = \{r^i s^j : i = 0, \ldots, 9,\ j = 0, 1;\ r^{10} = s^2 = e,\ sr = r^9 s\}.$$

The conjugacy classes of D_{10} are

$$\{e\},\ \{r^5\},\ \{r, r^9\},\ \{r^2, r^8\},\ \{r^3, r^7\},\ \{r^4, r^6\},$$
$$\{s, r^2s, r^4s, r^6s, r^8s\},\ \{rs, r^3s, r^5s, r^7s, r^9s\}.$$

For each of the following pairs H, K of subgroups of D_{10}, determine whether D_{10} is the internal direct product of H and K. Justify your answer fully. You can assume that H and K are subgroups of D_{10}.

(i) $H = \{e, r^2, r^4, r^6, r^8\}$, $K = \{e, r^5, s, r^5 s\}$

(ii) $H = \{e, s, r^2, r^4, r^6, r^8, r^2 s, r^4 s, r^6 s, r^8 s\}$, $K = \{e, r^5\}$ [4]

(b) Let p and q be distinct primes.

(i) Is $\mathbb{Z}_{p^2 q}$ isomorphic to $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$? Justify your answer.

(ii) Is $\mathbb{Z}_{p^2 q}$ isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$? Justify your answer. [4]

You may like to use this table for your answer to Question 11.

Group	Order of group	Isomorphic groups	Orders of elements	Cyclic? Abelian?
$\mathbb{Z}_{15}$	15	$\mathbb{Z}_3 \times \mathbb{Z}_5$	1, 3, 5, 15	Cyclic
$\mathbb{Z}_2 \times \mathbb{Z}_{10}$	20	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$	1, 2, 5, 10	Abelian, non-cyclic
D_{12}				
$\mathbb{Z}_3 \times \mathbb{Z}_4$				
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$				